EIGENFUNCTIONS OF MACDONALD'S q-DIFFERENCE OPERATOR FOR THE ROOT SYSTEM OF TYPE C_n

KATSUHISA MIMACHI

Abstract. We construct an integral representation of eigenfunctions for Macdonald's q-difference operator associated with the root system of type C_n . It is given in terms of a restriction of a q-Jordan-Pochhammer integral. Choosing a suitable cycle of the integral, we obtain an integral representation of a special case of the Macdonald polynomial for the root system of type C_n .

1. Introduction

Macdonald introduced the q-difference operators [5] to define his orthogonal polynomials associted with root sytems. In the case of a root system of type C_n , his q-difference operator is given by

$$E = \sum_{a_1,\cdots,a_n=\pm 1} \prod_{1\le i<j\le n} \frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}} \prod_{1\le i\le n} \frac{1-ty_i^{2a_i}}{1-y_i^{2a_i}} T_{y_i}^{\frac{a_i}{2}} ,$$

where

$$(T_{y_i}f)(y_1,\ldots,y_n) = f(y_1,\ldots,qy_i,\ldots,y_n) .$$

The present paper is devoted to study the eigenvalue problem associated with this operator E . In particular, we construct an integral representation, which is given by a restriction of a q-Jordan-Pochhammer integral, of eigenfunctions in some special cases. It turns out that, taking a suitable cycle, such an integral expresses the Macdonald polynomial of C_n type parametrized by the partition $(\lambda, 0, \ldots, 0)$. This representaion leads to a more explicit expression.

Here we recall the definition of the Macdonald polynomial $P_\mu(y|q,t)$ associated with the root system of type C_n . It is the eigenfunction of E with respect to the eigenvalue

$$c_\mu = q^{-\frac{1}{2}(\mu_1+\cdots+\mu_n)} \prod_{i=1}^{n} (1+q^{\mu_i}\, t^{n-i+1})$$

of the form

$$P_\mu(y|q,t) = m_\mu + \sum_{\nu<\mu} a_{\mu\,\nu} m_\nu ,$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a partition, a sequence of non-negative integers in decreasing order, $m_\mu = \sum_{\nu\in W(C_n)\mu} e^\nu$ with $W(C_n)$ the Weyl group of type C_n and $\nu < \mu$ is defined to be $\mu - \nu \in Q^+$ with Q^+ the positive cone of the root lattice.

Besides the A_{n-1} case, the solution of the eigenvalue problem for the Macdonald operator is not well studied (See [9, 10, 11] and [7]). We expect that this paper

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represents a step toward understanding the BC_n type Macdonald polynomials[3]. It is noteworthy that even in the classical (q=1) case was not previously known that such an integral gives spherical functions associated with the root system C_n. For related works on BC_n type spherical functions, we refer the reader to [2] and references therein.

Throughout this paper, q is regarded as a real number satisfying $0 < q < 1$, and $t = q^k$ where $k \in \mathbb{Z}_{\geq 1}$.

2. A restriction of a q-Jordan-Pochhammer integral

Let us introduce a 1-form

$$\begin{aligned}\Phi &= x^\lambda \prod_{1\leq j\leq n} \frac{(ty_j/x;q)_\infty \ (ty_j^{-1}/x;q)_\infty}{(\ y_j/x;q)_\infty \ (\ y_j^{-1}/x;q)_\infty} \frac{dx}{x} \\ &= x^\lambda \prod_{1\leq j\leq n} \frac{1}{(y_j/x;q)_k \ (y_j^{-1}/x;q)_k} \frac{dx}{x}\,,\end{aligned} \tag{2.1}$$

where $\lambda \in \mathbb{Z}_{\geq 0}$, $(a;q)_\infty = \prod_{i\geq 0}(1-aq^i)$ and $(a;q)_m = (a;q)_\infty/(q^m a;q)_\infty$. This can be regarded as a 1-form corresponding to a restriction of a q-Jordan-Pochhammer integral

$$x^\lambda \prod_{1\leq j\leq 2n} \frac{(ty_j/x;q)_\infty}{(y_j/x;q)_\infty} \frac{dx}{x}\,,$$

which is studied in [8] and [1].

Our first result is the following:

Theorem 1. *For any cycle $\mathcal{C}$, the function $\int_{\mathcal{C}} \Phi$ satisfies the equation*

$$E \int_{\mathcal{C}} \Phi = c_{(\lambda,0,\dots,0)} \int_{\mathcal{C}} \Phi\,.$$

This implies that linearly independent solutions are obtained by choosing several cycles. Indeed, if we put $C_i^{(+)}$ (or $C_i^{(-)}$) for each $i = 1,\dots,n$ to be a path with the counterclockwise direction so that the poles at $w = y_i, y_i q, \dots, y_i q^{k-1}$ (or $w = y_i^{-1}, y_i^{-1}q, \dots, y_i^{-1}q^{k-1}$, respectively) are inside the path and other poles from Φ are outside, we have the following rational solutions:

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}} \int_{C_i^{(+)}} \Phi &= y_i^\lambda \frac{1}{(q;q)_{k-1} \prod\limits_{\substack{1\leq j\leq n\\ j\neq i}} (y_j/y_i;q)_k \prod\limits_{1\leq j\leq n} (y_j^{-1}y_i^{-1};q)_k} \\ &\quad \times \sum_{l=0}^{k-1} \prod_{j=1}^{n} \frac{(t^{-1}qy_i/y_j;q)_l \ (t^{-1}qy_iy_j;q)_l}{(qy_i/y_j;q)_l \ (qy_iy_j;q)_l} (t^{2n}q^\lambda)^l\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}} \int_{C_i^{(-)}} \Phi &= y_i^{-\lambda} \frac{1}{(q;q)_{k-1} \prod\limits_{\substack{1\leq j\leq n\\ j\neq i}} (y_i/y_j;q)_k \prod\limits_{1\leq j\leq n} (y_jy_i;q)_k} \\ &\quad \times \sum_{l=0}^{k-1} \prod_{j=1}^{n} \frac{(t^{-1}qy_j/y_i;q)_l \ (t^{-1}qy_i^{-1}y_j^{-1};q)_l}{(qy_j/y_i;q)_l \ (qy_i^{-1}y_j^{-1};q)_l} (t^{2n}q^\lambda)^l\,.\end{aligned} \tag{2.3}$$

The calculation is carried out by means of the residue calculus.

Since λ is a non-negative integer, the sum of the pathes $\sum_{i=1}^n C_i^{(+)} + \sum_{i=1}^n C_i^{(-)}$ is homologous to a path C which circles the origin in the positive sense so that all poles from Φ are inside the path. The integral on this cycle C gives the Macdonald polynomial $P_{(\lambda,0,\dots,0)}(y|q,t)$.

Theorem 2. *If the cycle C is that above, we have*

$$\frac{1}{2\pi\sqrt{-1}}\int_C \Phi = \frac{(t;q)_\lambda}{(q;q)_\lambda} P_{(\lambda,0,\dots,0)}(y|q,t). \tag{2.4}$$

Moreover, applying the q-binomial theorem

$$\sum_{i\ge 0}\frac{(a;q)_i}{(q;q)_i} z^i = \frac{(az;q)_\infty}{(z;q)_\infty} \quad (|z|<1)$$

with the residue calculus to our integral, we obtain an exact expression of $P_{(\lambda,0,\dots,0)}(y|q,t)$.

Corollary .

$$P_{(\lambda,0,\dots,0)}(y|q,t) = \frac{(q;q)_\lambda}{(t;q)_\lambda} \sum_{\substack{i_1+\cdots+i_{2n}=\lambda \\ i_1,\dots,i_{2n}\ge 0}} \frac{(t;q)_{i_1}\cdots(t;q)_{i_{2n}}}{(q;q)_{i_1}\cdots(q;q)_{i_{2n}}} y_1^{i_1-i_{2n}} y_2^{i_2-i_{2n-1}} \cdots y_n^{i_n-i_{n+1}}.$$

3. Proof of Theorem 1

Lemma 1. *We have*

$$\sum_{a_1,\dots,a_n=\pm 1}\prod_{1\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{1\le i\le n}\frac{1-ty_i^{2a_i}}{1-y_i^{2a_i}} = \prod_{i=1}^n (1+t^i).$$

From the formula by Macdonald[4] about the Poincaré series of Coxter systems, we have

$$\sum_{w\in W(C_n)} w\left\{\prod_{1\le i<j\le n}\frac{1-ty_iy_j}{1-y_iy_j}\prod_{1\le i\le n}\frac{1-ty_i}{1-y_i}\right\} = \frac{\prod_{i=1}^n(1-t^{2i})}{(1-t)^n} \tag{3.1}$$

and

$$\sum_{w\in W(A_{n-1})} w\left\{\prod_{1\le i<j\le n}\frac{1-ty_i/y_j}{1-y_i/y_j}\right\} = \frac{\prod_{i=1}^n(1-t^i)}{(1-t)^n}. \tag{3.2}$$

Here $W(C_n)$ or $W(A_{n-1})$ denotes the Weyl group of the root system of type C_n or A_{n-1}, respectively. By applying the formula (3.2) to (3.1), we obtain

$$\begin{aligned}&\sum_{w\in W(C_n)} w\left\{\prod_{1\le i<j\le n}\frac{1-ty_iy_j}{1-y_iy_j}\prod_{1\le i\le n}\frac{1-ty_i}{1-y_i}\right\}\\ &= \frac{\prod_{i=1}^n(1-t^i)}{(1-t)^n}\sum_{a_1,\dots,a_n=\pm 1}\prod_{1\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{1\le i\le n}\frac{1-ty_i^{2a_i}}{1-y_i^{2a_i}}.\end{aligned}$$

Hence we derive the desired relation. □

Lemma 2.

$$\sum_{a_1,\dots,a_n=\pm 1}\prod_{1\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{1\le i\le n}\frac{(1-ty_i^{2a_i})(1-y_i^{a_i}/x)}{(1-y_i^{2a_i})(1-ty_i^{a_i}/x)} \tag{3.3}$$
$$=\prod_{i=1}^{n-1}(1+t^i)\left\{1+t^n\prod_{i=1}^{n}\frac{(1-y_i/x)(1-y_i^{-1}/x)}{(1-ty_i/x)(1-ty_i^{-1}/x)}\right\}.$$

Proof. We prove the desired equality by means of partial fraction decompositions. Firstly, let us take the residue of the left-hand side of (3.3) at $x=ty_1$:

$$\mathrm{Res}_{x=ty_1}\left\{\sum_{a_1,\dots,a_n=\pm 1}\prod_{1\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{1\le i\le n}\frac{(1-ty_i^{2a_i})(1-y_i^{a_i}/x)}{(1-y_i^{2a_i})(1-ty_i^{a_i}/x)}\right\}\frac{dx}{x}$$
$$=\frac{(1-t^{-1})(1-ty_1^2)}{1-y_1^2}\sum_{a_2,\dots,a_n=\pm 1}\prod_{2\le j\le n}\frac{1-ty_1y_j^{a_j}}{1-y_1y_j^{a_j}}$$
$$\times\prod_{2\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{2\le i\le n}\frac{(1-ty_i^{2a_i})(1-t^{-1}y_i^{a_i}/y_1)}{(1-y_i^{2a_i})(1-y_i^{a_i}/y_1)}$$
$$=t^{1-n}\frac{(1-t^{-1})(1-ty_1^2)}{1-y_1^2}\prod_{2\le j\le n}\frac{(1-ty_1y_j)(1-ty_1/y_j)}{(1-\ y_1y_j)(1-\ y_1/y_j)}$$
$$\times\sum_{a_2,\dots,a_n=\pm 1}\prod_{2\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-\ y_i^{a_i}y_j^{a_j}}\prod_{2\le i\le n}\frac{1-ty_i^{2a_i}}{1-\ y_i^{2a_i}}.$$

This is equal to

$$t^{1-n}\frac{(1-t^{-1})(1-ty_1^2)}{1-y_1^2}\prod_{i=1}^{n-1}(1+t^i)\prod_{2\le j\le n}\frac{(1-ty_1y_j)(1-ty_1/y_j)}{(1-\ y_1y_j)(1-\ y_1/y_j)}, \tag{3.4}$$

from Lemma 1.

Secondly, by noticing that

$$\mathrm{Res}_{x=ty_1}\prod_{1\le i\le n}\frac{(1-\ y_i/x)(1-\ y_i^{-1}/x)}{(1-ty_i/x)(1-ty_i^{-1}/x)}\frac{dx}{x}$$
$$=t^{1-2n}\frac{(1-t^{-1})(1-ty_1^2)}{1-y_1^2}\prod_{2\le j\le n}\frac{(1-ty_1y_j)(1-ty_1/y_j)}{(1-\ y_1y_j)(1-\ y_1/y_j)},$$

we know that the residue of the right-hand side of (3.3) at $x=ty_1$ is equal to (3.4). Hence, the symmetry of (3.3) with respect to the variables $y_1^{\pm 1},\dots,y_n^{\pm 1}$ leads to the fact that the residues of both sides of (3.3) at each $x=ty_i\quad(i=1,\dots,n)$ or $x=ty_i^{-1}\quad(i=1,\dots,n)$ are equal.

On the other hand, if x goes to ∞, the left-hand side of (3.3) tends to

$$\sum_{a_1,\dots,a_n=\pm 1}\prod_{1\le i<j\le n}\frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}}\prod_{1\le i\le n}\frac{1-ty_i^{2a_i}}{1-\ y_i^{2a_i}},$$

which is equal to $\prod_{i=1}^n(1+t^i)$ by Lemma 1, and the right-hand side of (3.3) tends also to $\prod_{i=1}^n(1+t^i)$. This completes the proof of Lemma 2. □

Let us proceed to prove our Theorem 1.

Note that

$$\prod_{1\le i\le n} T_{y_i}^{\frac{a_i}{2}} \int \Phi = \int_C x^\lambda \prod_{i=1}^n \frac{\left(q^{\frac{a_i}{2}}\frac{ty_i}{x}\right)_\infty \left(q^{-\frac{a_i}{2}}\frac{ty_i^{-1}}{x}\right)_\infty}{\left(q^{\frac{a_i}{2}}\frac{y_i}{x}\right)_\infty \left(q^{-\frac{a_i}{2}}\frac{y_i^{-1}}{x}\right)_\infty}\frac{dx}{x}$$

$$= q^{-\frac{\lambda}{2}} \int_C x^\lambda \prod_{i=1}^n \frac{\left(q^{\frac{1}{2}(1+a_i)}\frac{ty_i}{x}\right)_\infty \left(q^{\frac{1}{2}(1-a_i)}\frac{ty_i^{-1}}{x}\right)_\infty}{\left(q^{\frac{1}{2}(1+a_i)}\frac{y_i}{x}\right)_\infty \left(q^{\frac{1}{2}(1-a_i)}\frac{y_i^{-1}}{x}\right)_\infty}\frac{dx}{x}$$

$$= q^{-\frac{\lambda}{2}} \int_C \prod_{i=1}^n \frac{1-y_i^{a_i}/x}{1-ty_i^{a_i}/x}\Phi\,,$$

where the second equality is given by the change of integration variable such that $x \mapsto q^{-\frac{1}{2}}x$.

Therefore, by using Lemma 2, we obtain

$$E\int_C \Phi = q^{-\frac{\lambda}{2}} \int_C \left\{ \sum_{a_1,\dots,a_n=\pm 1} \prod_{1\le i<j\le n} \frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}} \prod_{1\le i\le n} \frac{(1-ty_i^{2a_i})(1-y_i^{a_i}/x)}{(1-y_i^{2a_i})(1-ty_i^{a_i}/x)} \right\}\Phi$$

$$= q^{-\frac{\lambda}{2}} \prod_{i=1}^{n-1}(1+t^i) \int_C \left\{ 1 + t^n \prod_{i=1}^n \frac{(1-y_i/x)(1-y_i^{-1}/x)}{(1-ty_i/x)(1-ty_i^{-1}/x)} \right\}\Phi$$

$$= (q^{-\frac{\lambda}{2}} + q^{\frac{\lambda}{2}}t^n) \prod_{i=1}^{n-1}(1+t^i) \int_C \Phi\,.$$

Here, to derive the third equality, we have used the relation

$$\int_C \Phi = q^{-\lambda} \int_C \prod_{i=1}^n \frac{(1-y_i/x)(1-y_i^{-1}/x)}{(1-ty_i/x)(1-ty_i^{-1}/x)}\Phi\,,$$

which is given by the change of integration variable such that $x \mapsto q^{-1}x$.

This completes the proof of Theorem 1.

References

[1] Aomoto, K., Kato, Y., Mimachi, K.: A solution of the Yang-Baxter equation as connection coefficients of a holonomic q-difference system. Internat. Math. Res. Notices **1992**, No.1 , 7-15

[2] Debiard, A. and Gaveau, B.: Integral formulas for the spherical polynomials of a root system of type BC_2. Jour. of Funct. Anal. **119**, 401-454 (1994)

[3] Koornwinder, T. H.: *Askey-Wilson polynomials for root systems of type BC,* in Hypergeometric functions on domains of positivity, Jack polynomials and applications, D.St.P.Richards(ed.), Contemp. Math. **138**, Amer. Math. Soc. (1992), pp.189-204

[4] Macdonald, I.G.: The Poincaré series of a Coxeter group, Math. Ann. **199**, 161-174 (1972)

[5] Macdonald, I.G.: *A new class of symmetric functions,* in Actes Séminaire Lotharingen, Publ. Inst. Rech. Math. Adv., Strasbourg, 1988, 131-171

[6] Macdonald, I.G.: Affine Hecke algebras and orthogonal polynomials. Séminaire BOURBAKI, 47ème année, 1994-95, $n^\circ 797$

[7] Macdonald, I.G.: *Symmetric Functions and Hall Polynomials* (Second Edition), Oxford Mathematical Monographs, Clarendon Press, Oxford, 1995.

[8] Mimachi, K.: Connection problem in holonomic q-difference system associated with a Jackson integral of Jordan-Pochhammer type. Nagoya Math. J. **116**, 149-161 (1989)

[9] Mimachi, K.: A solution to quantum Knizhnik-Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type, Duke Math. J. **85**, 635-658 (1996).
[10] Mimachi, K.: Rational solutions to eigenvalue problems of the Macdonald type, in preparation.
[11] Mimachi, K. and Noumi, M.: An integral representation of eigenfunctions for Macdonald's q-difference operators, Tôhoku Math. J. **49**, 517-525 (1997).

Katsuhisa Mimachi
Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812-81
Japan
E-mail address: mimachi@ math.kyushu-u.ac.jp